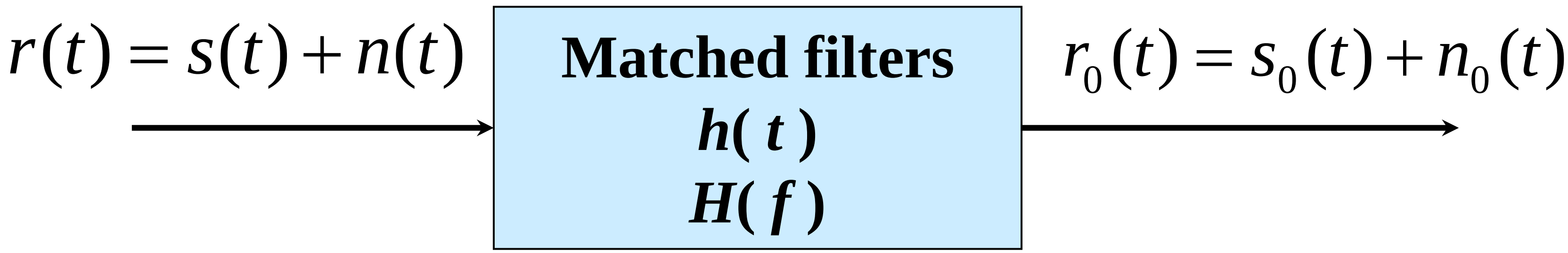
Chapter 6
Random Processes and Spectral Analysis

Matched filters

6-8 Matched filters

6-8 Matched filters

A general representation for a matched filter is illustrated as follows:



Matched filters design criterion:

To find $h(t)$ or, equivalently, $H(f)$, so that

$$\left(\frac{S}{N}\right)_{out} = \frac{s_0^2(t)}{\overline{n_0^2(t)}}$$

is a maximum at a sampling time $t=t_0$.

6-8 Matched filters

General results

$$r(t) = s(t) + n(t) \longrightarrow \boxed{\begin{matrix}\textbf{Matched filters}\\ \boldsymbol{h(t)}\\ \boldsymbol{H(f)}\end{matrix}} \longrightarrow r_0(t) = s_0(t) + n_0(t)$$

$$s_0(f) = H(f)s(f)$$

$$s_0(t_0) = \int_{-\infty}^{\infty} H(f)s(f)e^{j\omega t_0}df$$

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 p_n(f)df$$

$$\left(\frac{S}{N}\right)_{out} = \frac{s_0^2(t)}{\overline{n_0^2(t)}} = \frac{\left|\int_{-\infty}^{\infty} H(f)s(f)e^{j\omega t_0}df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 p_n(f)df}$$

General results

Theorem

The matched filter is the linear filter that maximizes

$$\left(\frac{S}{N}\right)_{out} = \frac{\overline{s_0^2(t)}}{\overline{n_0^2(t)}}$$

And that has a transfer function given by

$$H(f) = K\frac{S^*(f)}{p_n(f)}e^{-j\omega t_0}$$

6-8 Matched filters

Results for white noise

Theorem

When the input noise is white, the impulse response of the matched filter becomes

$$h(t) = Cs(t_0 - t)$$

Where

C is an arbitrary real positive constant, t_0 is the time of the peak signal output, $s(t)$ is the known input–signal waveshape.

6-8 Matched filters

Results for white noise

An important property of matched filters:

the actual value of $(S/N)_{out}$ can be obtained form the matched filter.

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{p_n(f)} df$$

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_0/2} df = \frac{2}{N_0}\int_{-\infty}^{\infty} |S(f)|^2 df = \frac{2}{N_0}\int_{-\infty}^{\infty} s^2(t) dt$$

$$\left(\frac{S}{N}\right)_{out} = \frac{2E_s}{N_0}$$

The result states that $(S/N)_{out}$ depends on the signal energy and PSD level of the noise, and not on the particular signal waveshape that is used.

Matched filter realization

(1) Integrate-and-dump matched filter

Example 6-11

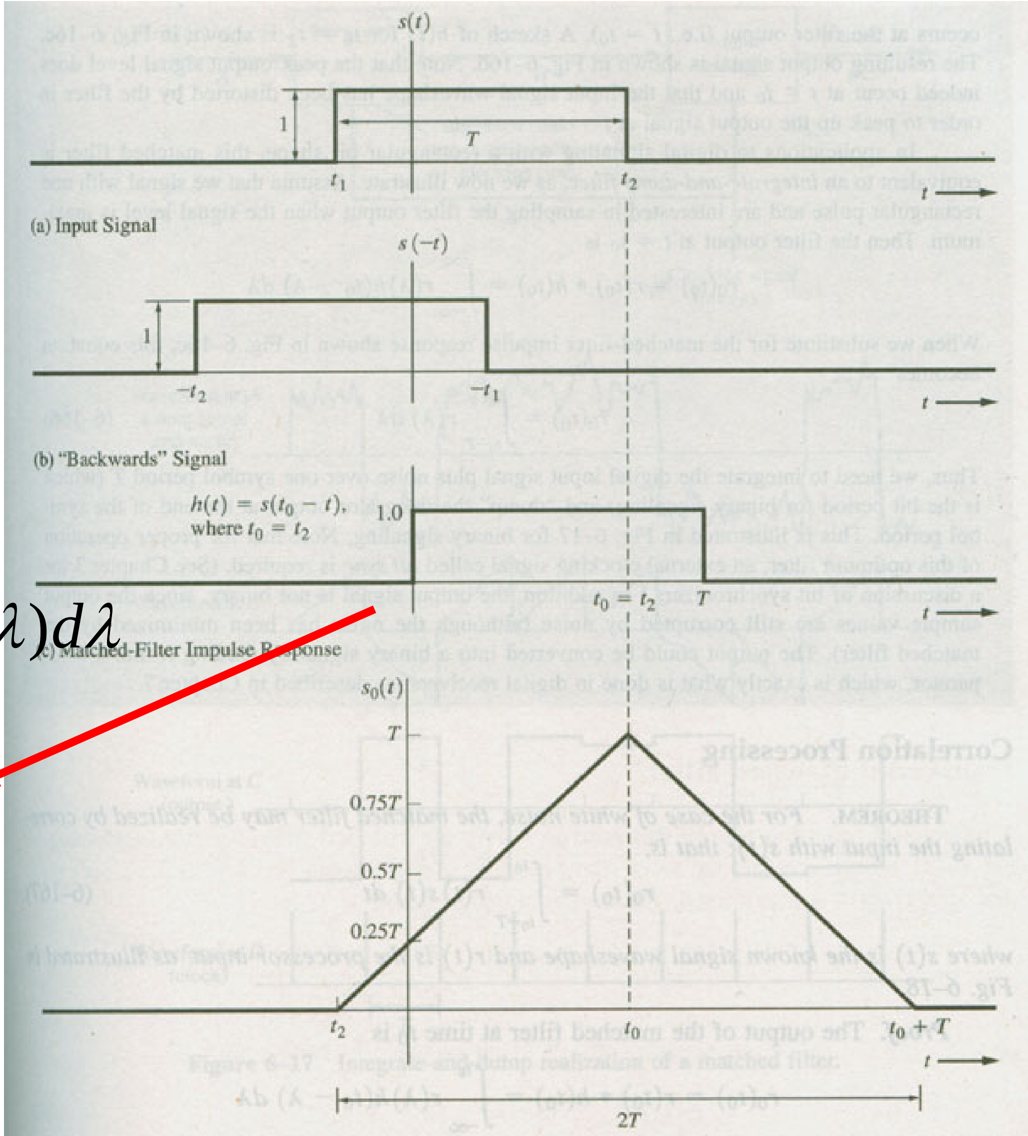
$$s(t)=\begin{cases}1\ , & t_1 \le t \le t_2 \\ 0\ , & others\end{cases}$$

$$h(t)=s(t_0-t)=s(-(t-t_0))$$

$$r_0(t_0)=r(t_0)*h(t_0)=\int_{-\infty}^{\infty} r(\lambda)h(t_0-\lambda)d\lambda$$

$$h(t)=\begin{cases}s(t_0-t)=1, & 0 \le t \le T \\ 0\ , & others\end{cases}$$

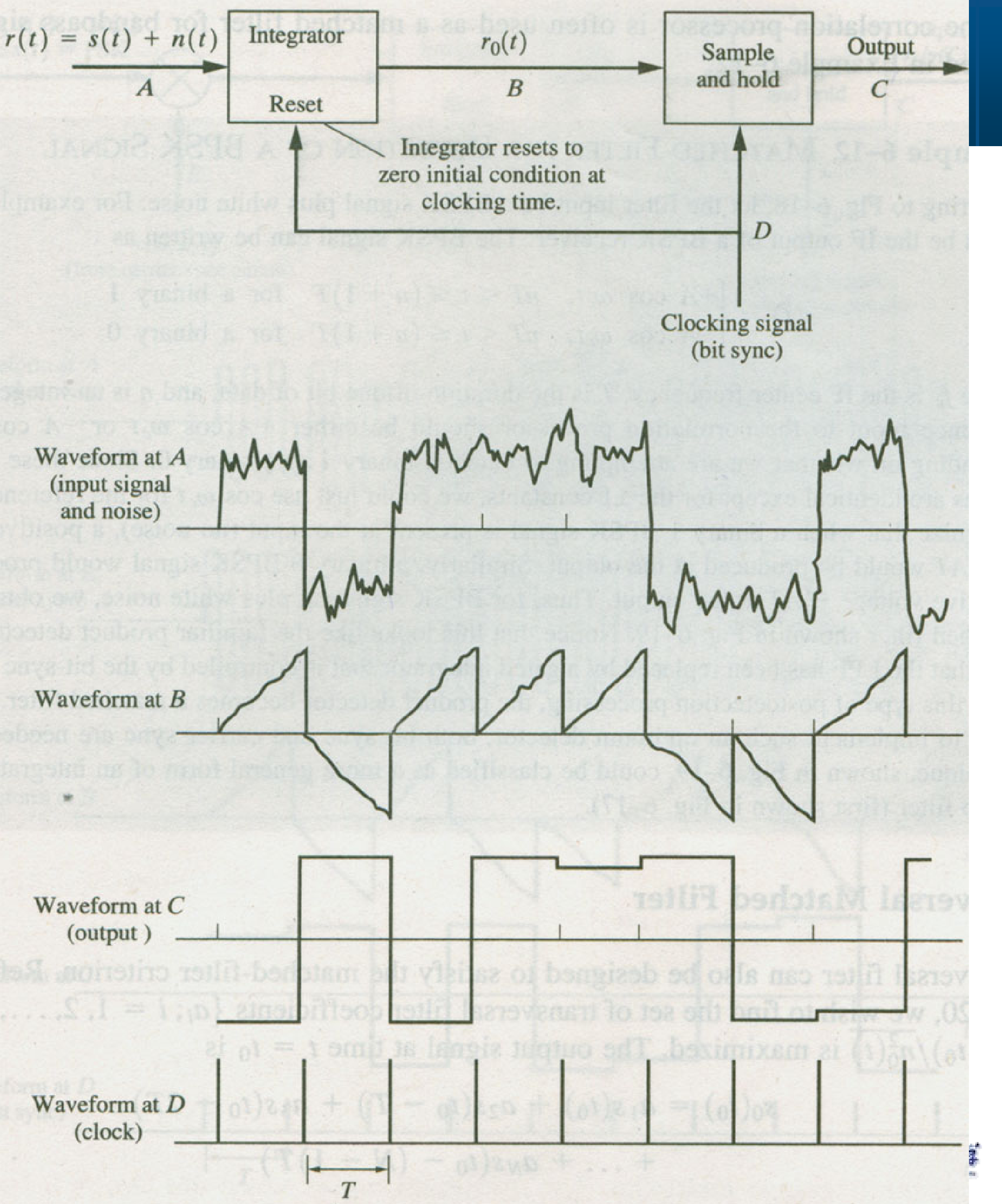
$$r_0(t_0)=\int_{t_0-T}^{t_0} r(\lambda)d\lambda$$



Matched filter realization

(1) Integrate-and-dump matched filter

Example 6-11



Matched filter realization

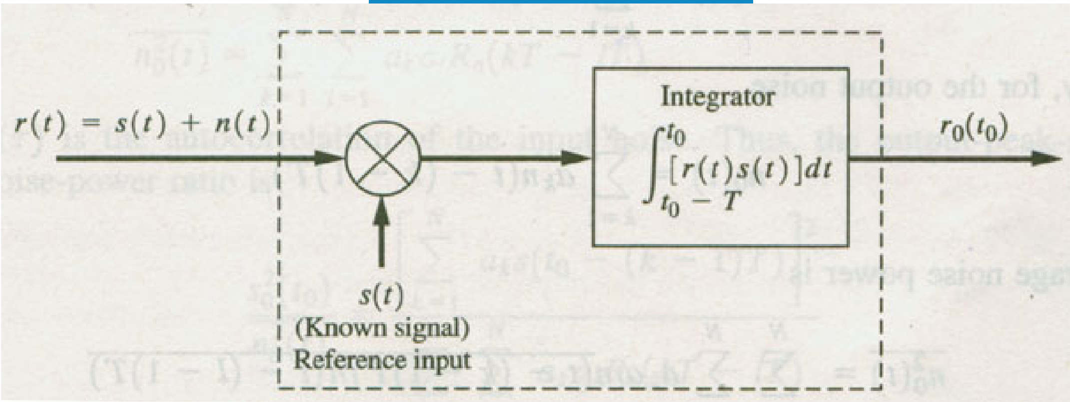
(2) Correlation processing

For the case of white noise

$$r_0(t_0) = r(t_0) * h(t_0) = \int_{-\infty}^{\infty} r(\lambda) h(t_0 - \lambda) d\lambda$$

$$h(t) = \begin{cases} s(t_0 - t) & 0 \le t \le T \\ 0 & others \end{cases}$$

$$r_0(t_0) = \int_{t_0 - T}^{t_0} r(\lambda) s[t_0 - (t_0 - \lambda)] d\lambda = \int_{t_0 - T}^{t_0} r(t) s(t) dt$$



(2) Correlation processing

Theorem:

For the case of white noise, the matched filter may be realized by correlating the input *r*(*t*) with *s*(*t*), that is,

$$r_0(t_0) = \int_{t_0-T}^{t_0} r(t)s(t)dt$$

Where *s*(*t*) is the known signal waveshape and *r*(*t*) is the processor input.

Matched filter realization

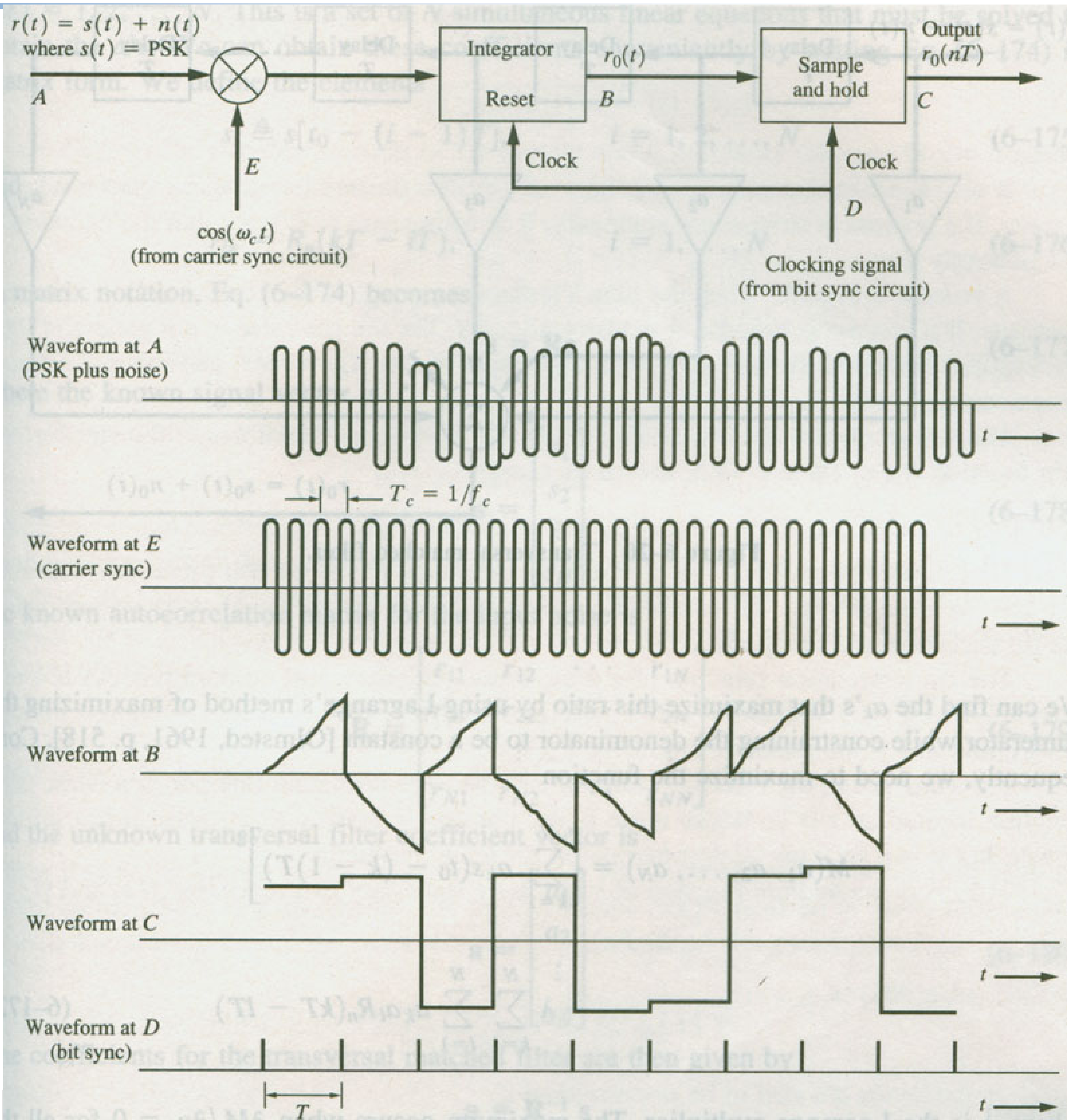
Example 6-12

Matched filter for detection of a BPSK signal

$$s(t) = \begin{cases} +A\cos\omega_c t & \textit{for "1"} \\ -A\cos\omega_c t & \textit{for "0"} \end{cases}$$

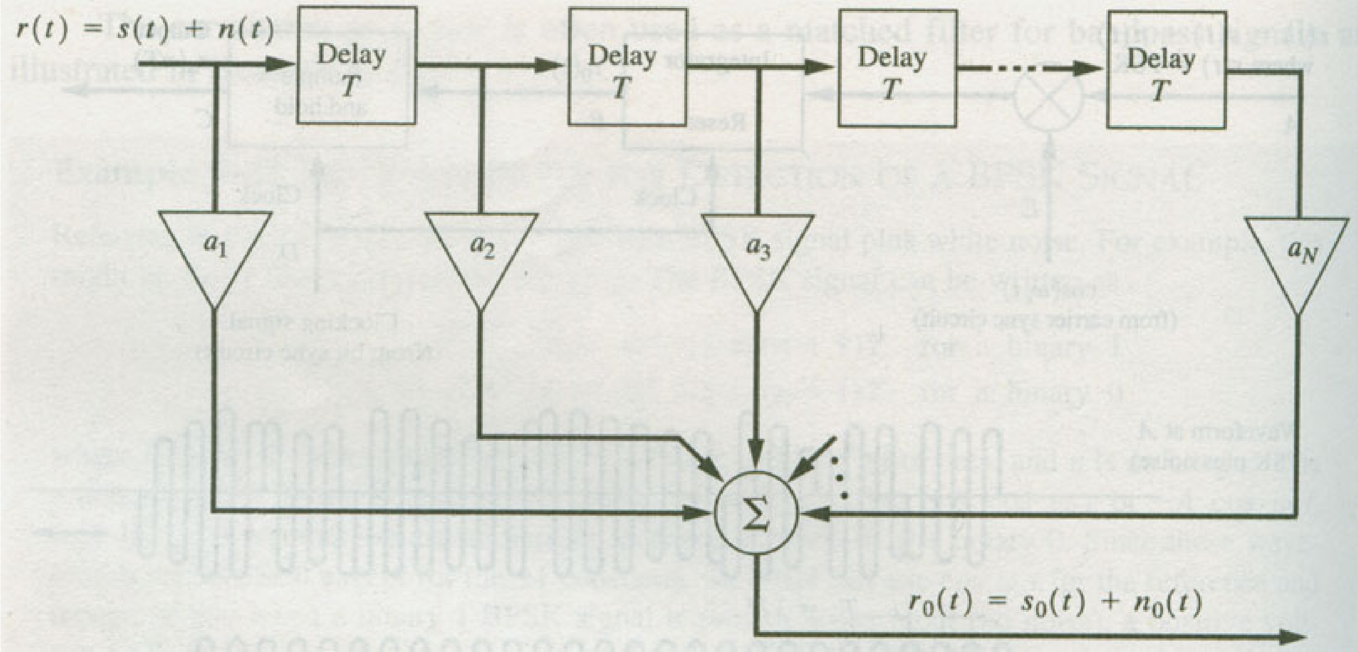
$$nT \le t \le (n+1)T$$

$$r_0(t_0) = \int_{t_0-T}^{t_0} r(t)s(t)dt$$



(3) Transversal matched filter

A transversal filter can be designed to satisfy the matched-filter criterion.



Design goal:

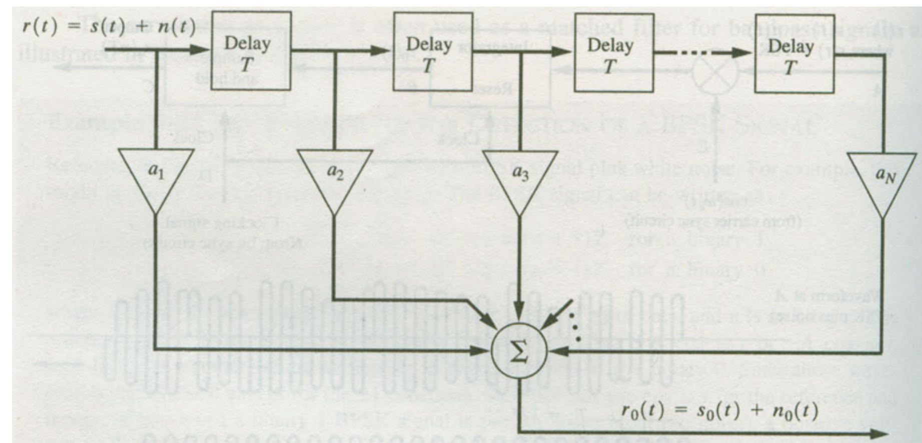
To find the set of transversal filter coefficients $\{a_i,\ i=1, 2, \ldots, N\}$ such that $s_0^2(t_0)/\overline{n_0^2(t)}$ is maximized

Matched filter realization

(3) Transversal matched filter

$$s_0(t_0) = a_1 s(t_0) + a_2 s(t_0 - T) + a_3 s(t_0 - 2T) + \cdots\cdots + a_N s[t_0 - (N-1)T]$$

$$= \sum_{k=1}^{N} a_k s(t_0 - (k-1)T)$$



$$n_0(t) = \sum_{k=1}^{N} a_k n(t - (k-1)T)$$

$$\overline{n_0^2(t)} = \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l \overline{n(t-(k-1)T)\; n(t-(l-1)T)}$$

$$\frac{s_0^2(t_0)}{\overline{n_0^2(t)}} = \frac{\left[\sum_{k=1}^{N} a_k s(t_0 - (k-1)T)\right]^2}{\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l R_n(kT - lT)}$$

Matched filter realization

$$\frac{s_0^2(t_0)}{\overline{n_0^2(t)}} = \frac{\left[\sum_{k=1}^{N} a_k s(t_0-(k-1)T)\right]^2}{\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l R_n(kT-lT)}$$

Using Lagrange's method of maximizing the numerator

$$M(a_1,a_2\cdots a_N) = \left[\sum_{k=1}^{N} a_k s(t_0-(k-1)T)\right]^2 - \lambda \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l R_n(kT-lT)$$

$$\frac{\partial M}{\partial a_i} = 0 \Rightarrow \frac{\partial M}{\partial a_i} = 2\left[\sum_{k=1}^{N} a_k s(t_0-(k-1)T)\right] s(t_0-(i-1)T) - 2\lambda \sum_{k=1}^{N} a_k R_n(kT-iT) = 0$$

$$i = 1,2\cdots\cdots N$$

Because: $\sum_{k=1}^{N} a_k s(t_0-(k-1)T) = s_0(t_0)$

Furthermore, let: $\lambda = s_0(t_0)$ $\Rightarrow$

$$s(t_0-(i-1)T) = \sum_{k=1}^{N} a_k R_n(kT-iT)$$

Matched filter realization

We define:

$$s_i = s(t_0 - (i-1)T)$$

$$r_{ik} = R_n(kT - iT)$$

$$i = 1,2\cdots\cdots N$$

$$s(t_0 - (i-1)T) = \sum_{k=1}^{N} a_k R_n(kT - iT) \Rightarrow s = Ra$$

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} \quad R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{bmatrix} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

The coefficients for transversal matched filter are given by

$$a = R^{-1}s$$

Summary

- **matched filter is a linear filter that maximizes the instantaneous output signal power for a given input signal waveshape**

- **For the case of white noise, the impulse response of the matched filter is** $h(t) = Cs(t_0 - t)$

- **The matched filter can be realized in many forms, Such as the integrate–and–dump, the correlator, and the transversal filter**